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Geometric phase in the causal quantum theories

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Abstract

The projective-geometric formulation of geometric phase for any ensemble in the causal quantum theories is given. This formulation generalizes the standard formulation of geometric phase to any causal ensemble including the cases of a single causal trajectory, the experimental geometric phase and the classical geometric phase. © 1997 Elsevier Science B.V.

The geometric phase is mathematically a property of any complex inner product space. This kinematic formulation has been developed by Aitchison and Wanclik [1] and Mukunda and Simon [2]. Hilbert space being central to standard quantum theory, the quantum geometric phase has great relevance. From the kinematic approach, the important cases for quantum theory of the Samuel–Bhandari [3] non-cyclic geometric phase, the Aharonov–Anandan [4] geometric phase for cyclic motion and the Berry [5] geometric phase for adiabatic cyclic motion have been shown to be particular cases [1,2]. The quantum geometric phase has also been semiclassically related by Berry [6] to the classical Hannay [7] angle defined only for integrable systems. A difficulty arises in the standard formulation of the quantum geometric phase by the inability to define the geometric phase for arbitrary subensembles. This could be overcome in the causal quantum theories [8–12]. However, the Hilbert space structure is secondary in these theories and they might appear at first sight to be unable to accommodate general definitions of geometric phase.

In this Letter we show that the causal theories and therefore classical mechanics in their classical limit allow a projective-geometric formulation and therefore the causal geometric phase can be obtained from the general kinematic approach to geometric phases. A general formulation is presented that generalizes the standard quantum geometric phase to a geometric phase valid for any causal quantum ensemble. Recently, Parmenter and Valentine [13] have proposed, by analogy with the cyclic geometric phase, a phase for a special class of Bohmian trajectories but it remains open in their approach if this phase is projective-geometric in the same sense as the standard geometric phase and unique with respect to the standard geometric phase. Sjöqvist and Carlsen [14] have obtained the geometric phase for the decoherent wavefunction. In this Letter we show that these two cases and their generalizations as well as other cases can be obtained from a single general formulation that is geometric from the outset and unique with respect to the standard geometric phase.

We have organised this Letter as follows. We briefly discuss the kinematic approach to geometric phases. Secondly we obtain the general expression

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for the geometric phase for arbitrary causal ensembles. Then we show how previous results in the literature can be obtained from our general expression along with their generalizations and new results. As particular cases, we obtain the standard quantum geometric phase, the geometric phase for a single causal trajectory, the geometric phase for experimental situations, the geometric phase for initially structureless ensembles and the classical geometric phase.

The geometric phase is a property of the path $\varphi(s)$ of the complex inner product space I. The two properties that define the geometric phase are (a) *reparametrization invariance*, invariance under the transformation $s \to \xi(s)$ with ξ a smooth monotonic function of s, which means that it is independent of the rate of change with s at which the path is traversed and (b) *global phase invariance*, invariance under the transformation $\varphi(s) \to z(s)\varphi(s)$ with $z(s)$ an arbitrary nonzero smooth $\mathbb{C}^*$-valued function of s. We denote the corresponding projection map by $\Pi : I \to \wp$ defined by $\Pi(z(s)\varphi(s)) = \Pi(\varphi(s))$. Thus the complex inner product space curve $X = \{\varphi(s) | s \in [s_1, s_2]\}$ projects on to the smooth projective complex inner product space $\wp$ curve $C = \Pi(X)$. The real geometric phase obeying the above two properties is a functional of the curve C of the form [1,2]

$$\gamma_g[C] = \arg(\langle \varphi(s_1) | \varphi(s_2) \rangle) - \operatorname{Im} \int_{s_1}^{s_2} \mathrm{d}s \left\langle \frac{\varphi(s)}{\|\varphi(s)\|} \middle| \frac{\mathrm{d}}{\mathrm{d}s} \middle| \frac{\varphi(s)}{\|\varphi(s)\|} \right\rangle. \tag{1}$$

The geometric phase for a particular theory is obtained with (1) and the evolution of the vector φ. In the following we consider the causal quantum theories. The state of a system of N particles is given in the causal quantum theories by the trajectories $\boldsymbol{d}_i$ of the N particles and by the guiding field $\Psi(d_1, d_2, \dots, d_N, t) = |\Psi| \exp(\mathrm{i}S/\hbar)$ solution of the Schrödinger equation. We consider the geometric phase for a causal ensemble P that obeys the continuity equation

$$\frac{\partial P}{\partial t} + \nabla \cdot (\dot{\boldsymbol{d}} P) = 0. \tag{2}$$

The geometric phase for the vector $\varphi = P^{1/2} \exp(\mathrm{i}S/\hbar)$ with the parameter s being the time t in (1) is of the form

$$\gamma_g[C] = \arg(\langle P^{1/2}(0) P^{1/2}(t) \exp(\mathrm{i}\Delta S/\hbar) \rangle) - \frac{1}{\hbar} \int_0^t \mathrm{d}t' \left\langle \frac{P(t')}{\langle P(t') \rangle} \partial S/\partial t \right\rangle, \tag{3}$$

where $\langle \ \rangle$ denotes integration over the space of definition of the causal distribution P. Expression (3) is the geometric phase for any arbitrary causal ensemble.

To obtain the geometric phase associated with a particular causal ensemble P, the form of the ensemble has to be substituted in (3). For the causal ensemble being the full quantum ensemble, $P = |\Psi|^2$, (3) reduces to the standard quantum geometric phase that numerically coincides with the definition of Samuel and Bhandari [3] for noncyclic evolution, the Aharonov–Anandan [4] geometric phase for cyclic motion and to the Berry [5] phase for adiabatic cyclic motion.

In the following we derive the geometric phase associated with a cyclic Bohmian trajectory $\boldsymbol{q}(t)$ which is a solution of $\dot{\boldsymbol{q}} = \nabla S/m|_{\boldsymbol{q}=\boldsymbol{q}(t)}$ with period T. For the ensemble P in (3) of the form $\delta(\boldsymbol{q} - \boldsymbol{q}(T)) = \delta(\boldsymbol{q} - \boldsymbol{q}(0))$ and using $\mathrm{d}S/\mathrm{d}t = \partial S/\partial t + \dot{\boldsymbol{q}} \cdot \nabla S$ expression (3) reduces to

$$\gamma_g[C] = \frac{m}{\hbar} \oint_C \dot{\boldsymbol{q}} \cdot \mathrm{d}\boldsymbol{q} = \frac{m}{\hbar} \int_0^T \dot{\boldsymbol{q}}^2 \mathrm{d}t. \tag{4}$$

Parmenter and Valentine [13] proposed (4) by analogy with the standard cyclic geometric phase for the particular class of cyclic Bohmian trajectories (a) with period $T = n\tau$ with n an integer, (b) associated with a cyclic wavefunction with period τ, $\Psi(\tau) = \Psi(0)\mathrm{e}^{\mathrm{i}z}$ with z a real number and (c) the wavefunction is made up of discrete eigenstates with commensurate frequencies. Moreover, in their approach (d) it is not demonstrated that the phase in (4) is projective-geometric and unique with respect to the standard quantum geometric phase and (e) they speculate that it is related in general with adiabaticity. It is clear in our derivation that (4) is more general as restrictions (a)-(c) are unnecessary. Our derivation has also the advantages of being geometric from the outset and unique with respect to the standard geometric phase as it is also obtained from the same formulation. Concerning the role of adiabaticity, it is true that the geometric phase was first discovered by

Berry [5] in the context of adiabatic evolution but subsequent generalizations [1–4] showed that adiabaticity need not play any role. Our derivation of the causal geometric phase has those kinematic approaches as a starting point and therefore adiabaticity is not a necessary condition for its existence.

Another interesting case is that of experimental situations which are treated in the causal theories as any other physical interactions. A detailed treatment of the causal description of measurements can be found in Refs. [8–10]. After the interaction has taken place, the apparatus particle enters a particular packet and as far as the particle is concerned the other packets can be ignored. The wavefunction after the interaction can be writen as

$$\Psi(x,y,t) = \psi(x,t)\Phi(y,t) + \Psi^{\perp}(x,y,t), \qquad (5)$$

with x the coordinate of the *system* and y that of the *environment*. Φ and $\Psi^{\perp}$ have disjoint y support and the actual *environmental* configuration $y(t) = Y(t)$ belongs to the support of Φ. The effective wavefunction for this case has thus the form

$$\Psi_Y = \psi\Phi(y = Y(t)). \qquad (6)$$

The causal theories permit a full analysis of the whole experimental situation beyond (5) and (6) for the general case

$$\Psi_Y(t) = \Psi(x,y = Y(t),t). \qquad (7)$$

This is the wavefunction for the *system* x conditioned to the *environment* $y = Y(t)$. The geometric phase for the relevant ensemble in the general experimental situation is then obtained substituting the effective distribution $P = |\Psi_Y|^2$ for the wavefunction (7) in the general expresion (3) and expression (8) of Sjöqvist and Carlsen [14] is obtained.

We now discuss the case of initially structureless causal ensembles obeying

$$\bar{P}(x,0) \equiv (\delta V)^{-1}\int_{\delta V} dx\, P(x,0) = P(x,0)$$

and $\overline{|\Psi(x,0)|^2} = |\Psi(x,0)|^2$. In a way analogous to classical statistical mechanics, Valentini [15] has obtained an H-theorem in Bohmian mechanics to show that $\bar{S} \le 0$ with $\bar{S} = -\int dx \bar{P} \ln(\bar{P}/\overline{|\Psi|^2})$ reaching its maximum $\bar{S} = 0$ for $\bar{P} = \overline{|\Psi|^2}$. This suggests that $\bar{P} \to |\Psi|^2$ and therefore the geometric phase of the initial structureless distribution would tend to that of the full quantum distribution in the coarse-grained sense, $\gamma_g[\bar{P}] \to \gamma_g[\overline{|\Psi|^2}]$. Two differences between the derivation of Valentini and that of classical statistical mechanics are that Bohmian mechanics is defined in position space and not in phase space and that the derivation breaks down for stationary states. These two differences can be removed in a causal quantum theory obtained from the coherent state representation of the Schrödinger equation we have recently proposed [12] and that we discuss in the following to study the classical limit of the quantum geometric phase.

Bohmian mechanics has an ontology close to that of classical mechanics and it might be expected that the classical geometric angle and even a classical geometric phase can be obtained in the classical limit. It is possible to obtain a position space classical geometric phase by using (3) for ensembles P for which the classical limit condition

$$-\frac{\hbar^2}{2m}\frac{\nabla^2|\Psi|}{|\Psi|} = 0$$

holds. However, no direct relation with the classical results is obtained as Bohmian mechanics is a position space theory and its classical limit is therefore a position space classical mechanics. Direct contact with the classical geometric phase and angle is obtained from a causal quantum theory in phase space we have recently obtained [12] from the coherent state representation of the Schrödinger equation [16–18]

$$i\hbar\frac{\partial}{\partial t}\langle q,p|\Psi\rangle = \langle q,p|\hat{H}(\hat{Q},\hat{P})|\Psi\rangle \qquad (8)$$

for a Hamiltonian of the general form $H = P^2/2m + V(Q)$ and

$$\langle q,p|\hat{Q}|\Psi\rangle = (q + i\hbar\nabla_p)\langle q,p|\Psi\rangle, \qquad (9)$$

$$\langle q,p|\hat{P}|\Psi\rangle = -i\hbar\nabla_q\langle q,p|\Psi\rangle. \qquad (10)$$

We substitute the polar form

$$\Psi(q,p,t) = R(q,p,t)\exp[(i/\hbar)S(q,p,t)]$$

for the wavefunction in (8). We consider the expansion of the potential of the general form $V(Q)=\sum_n a_n Q^n$ and we use the binomial expansion

$$(q+\mathrm{i}\hbar\nabla_p)^n = \sum_{r=0}^{n} \frac{n!}{r!(n-r!)} q^{n-r}(\mathrm{i}\hbar\nabla_p)^r \equiv \sum_{r=0}^{n} \hat{B}_r .$$

Separating real and imaginary parts, the following two equations are obtained

$$-\frac{\partial S}{\partial t} = \frac{(\nabla_q S)^2}{2m} + V + W, \tag{11}$$

$$\frac{\partial R^2}{\partial t} + \nabla_q\left[\frac{\nabla_q S}{m}R^2\right] + \nabla_p\left[(-\nabla_q V + Z')R^2\right] = 0, \tag{12}$$

with

$$W(q,p,t) = -\frac{\hbar^2}{2m}\frac{\nabla_q^2 R}{R} + \sum_n \frac{a_n}{R}\mathrm{Re}\left[\mathrm{e}^{-(\mathrm{i}/\hbar)S}\left(\sum_{r=1}^{n}\hat{B}_r\right)R\mathrm{e}^{(\mathrm{i}/\hbar)S}\right], \tag{13}$$

$$\begin{aligned} Z(q,p,t) &= \nabla_p(Z'R^2) \\ &= -\frac{2}{\hbar}\sum_n a_n \,\mathrm{Im}\left[R\mathrm{e}^{-(\mathrm{i}/\hbar)S}\left(\sum_{r=2}^{n}\hat{B}_r\right)R\mathrm{e}^{(\mathrm{i}/\hbar)S}\right]. \end{aligned} \tag{14}$$

Eq. (11) is a modified Hamilton–Jacobi equation with a new term W and Eq. (12) is a continuity equation in phase space. Eqs. (11)–(14) can be generalized to more dimensions and many particles. The causal trajectories are solution of

$$\dot{q} = \frac{\nabla_q S}{m}, \tag{15}$$

$$\dot{p} = -\nabla_q V(q) + Z'(q,p). \tag{16}$$

Analogous results to those presented in this Letter for Bohmian mechanics are obtained for this causal theory but in this case with a phase space structure and therefore $\mathrm{d}S/\mathrm{d}t = \partial S/\partial t + \dot{\boldsymbol{q}}\cdot\nabla_q S + \dot{\boldsymbol{p}}\cdot\nabla_p S$. The classical limit of Eqs. (11) and (12) is obtained for $W=Z=0$ as they reduce to the classical Hamilton–Jacobi equation and to the classical Liouville equation, respectively. For the classical limit, the equations of motion for the particle (15) and (16) reduce to Hamilton's equations of motion. Note that the classical limit might be obtained only for a subensemble P of the quantum ensemble and for certain intervals of time. The geometric phase in (3) for the classical limit has the following form in terms of the classical density, the classical action S and the Hamiltonian H,

$$\gamma_g[C] = \arg(\langle P^{1/2}(0)\,P^{1/2}(t)\exp(\mathrm{i}\Delta S/\hbar)\rangle) + \frac{1}{\hbar}\int_0^t \mathrm{d}t' \left\langle \frac{P(t')}{\langle P(t')\rangle} H\right\rangle, \tag{17}$$

where $\langle\ \rangle$ denotes integration over phase space and

$$\Delta S = \int_0^t \boldsymbol{p}\cdot\dot{\boldsymbol{q}}\,\mathrm{d}t - \int_0^t H(\boldsymbol{q},\boldsymbol{p},t)\,\mathrm{d}t.$$

For a cyclic classical trajectory $\boldsymbol{x}(t)=(\boldsymbol{q}(t),\boldsymbol{p}(t))$, with density obeying $\delta(\boldsymbol{x}-\boldsymbol{x}(T))=\delta(\boldsymbol{x}-\boldsymbol{x}(0))$, the geometric phase in (17) reduces to

$$\gamma_g = \frac{1}{\hbar}\oint_C \boldsymbol{p}\cdot\mathrm{d}\boldsymbol{q} = \frac{1}{\hbar}\iint_S \mathrm{d}\boldsymbol{p}\wedge\cdot\,\mathrm{d}\boldsymbol{q}. \tag{18}$$

The geometric phase in (17) allows one to extend the geometric phase from quantum mechanics to classical mechanics and it is valid for nonlinear and nonintegrable systems, including chaotic motion. Calculations for a nonintegrable system are reported elsewhere [19].

To conclude, we have obtained the geometric phase for any quantum causal ensemble in a general projective-geometric framework. We have derived as particular cases the standard quantum geometric phase, the geometric phase for individual causal trajectories, the geometric phase for experimental situations, the geometric phase for structureless ensembles and the classical geometric phase and angle.

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