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Galilean invariant structure of geometric phase

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Abstract

It is shown that the Galilean invariant structure of geometric phase in nonrelativistic quantum mechanics corresponds to the difference between geometric phases for two paths in ray space with common initial and final points. © 1997 Elsevier Science B.V.

The geometric phase has been extensively analysed and experimentally extracted in quantum systems. Of course, these experimental results can be explained by the Schrödinger equation alone. It is reasonable then to question how fundamental the geometric phase is. A necessary condition for a quantity to be a physical property of a system is to be Galilean invariant. *Is the geometric phase Galilean invariant?* is precisely the question that Sjöqvist, Brown and Carlsen (SBC) [1] have recently formulated in an illuminating analysis. Their answer is conclusive: geometric phase is not in general Galilean invariant. They also study existing experiments and conclude that they are interpreted using *effective* Hilbert spaces which are immune to passive Galilean boosts. Although for these experiments the geometric phase is an *effective* physical property, the SBC result challenges its physical significance.

In this Letter we ask a different question: *which is the quantity related to geometric phases that is Galilean invariant?* We first analyse the SBC result and we give afterwards the Galilean invariant structure of geometric phase.

We adopt here the Aitchison–Wanelik–Mukunda–Simon [2,3] kinematic approach to geometric phase. The geometric phase is the quantity which obeys reparametrization and global phase invariances of the form

$$\gamma_g[C] = \arg(\langle \Psi(t_i) | \Psi(t_f) \rangle) - \mathrm{Im} \int_{t_i}^{t_f} \mathrm{d}t \langle \Psi(t) | \dot{\Psi}(t) \rangle, \qquad (1)$$

with C a path in ray space $\mathscr{P}$ and t_i and t_f the times labelling the initial and final points of C. The above general formulation reduces to the Aharonov–Anandan [4] geometric phase γ_g^{AA} for cyclic evolution, itself a generalization of the Berry adiabatic and cyclic geometric phase [5]. As the closure property of a curve in $\mathscr{P}$ is frame dependent, it is necessary to study the Galilean invariance of the general noncyclic geometric phase (1). Consider the two inertial frames S and $\tilde{S}$, the motion of $\tilde{S}$ of velocity v relative to S and, say, parallel to the x-axis. The

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geometric phase (1) transforms under this subgroup of Galilean boosts as [1]

$$\gamma_g[\tilde{C}] = \gamma_g[C] + F. \quad (2)$$

Choosing the gauge such that the x-component of the vector potential that may possibly appear in the Hamiltonian is $A_x(Q, t) = 0$ for all the path C, the function F can be written as

$$F = \arg\left[\frac{\langle\Psi(t_i)\,|\,e^{ivP_x(t_f - t_i)/\hbar}\,|\,\Psi(t_f)\rangle}{\langle\Psi(t_i)\,|\,\Psi(t_f)\rangle}\right] - \frac{mv}{\hbar}\big(\langle\Psi(t_f)\,|\,Q_x\,|\,\Psi(t_f)\rangle - \langle\Psi(t_i)\,|\,Q_x\,|\,\Psi(t_i)\rangle\big). \quad (3)$$

Interestingly the function F above that breaks the Galilean invariance of the geometric phase in (2) is not a function of all the curve C in $\mathscr{P}$ but only a function of its initial and final points, $F = F(C(t_i), C(t_f), v)$. Now consider the geometric phases for two different curves C_1 and C_2. We can write an expression like (2) for both curves in $\mathscr{P}$. In the case these two curves have common initial and final points in $\mathscr{P}$ their corresponding F's in (3) are identical and it follows that

$$\gamma_g[C_1 - C_2] \equiv \gamma_g[C_1] - \gamma_g[C_2] = \gamma_g[\tilde{C}_1] - \gamma_g[\tilde{C}_2]$$
$$\text{for } C_1(t_i) = C_2(t'_i) \text{ and } C_1(t_f) = C_2(t'_f), \quad (4)$$

with t and t' the times labelling the points of curves C_1 and C_2, respectively. Therefore, although neither of the geometric phases $\gamma_g[C_1]$ and $\gamma_g[C_2]$ are Galilean invariant separately, their difference $\gamma_g[C_1 - C_2]$ is Galilean invariant. Relation (4) is our result: the Galilean invariant structure of the geometric phase is the difference between the geometric phases for two paths in $\mathscr{P}$ that have common initial and final points. This is illustrated in Fig. 1 for a system under a time dependent Hamiltonian that follows a closed path C in $\mathscr{P}$ for a frame S. This path can be divided in the paths C_1 and C_2, $C = C_1 + C_2$, that have common initial and final points. The geometric phases associated with a closed path in $\mathscr{P}$ are proportional to the area inside the path [4] (in the general noncyclic case it is proportional to the area inside the path and its geodesic closure [6]). Therefore for this case the areas inside C_1 and C_2 change

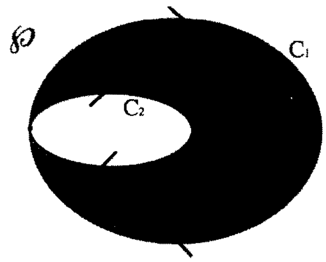

Fig. 1. Illustrative example of the Galilean invariant structure of the geometric phase in ray space $\mathscr{P}$. The system undergoes the cyclic path $C = C_1 + C_2$ in the frame S. The geometric phases for C_1 and C_2, $\gamma_g[C_1]$ and $\gamma_g[C_2]$ (the areas inside C_1 and C_2), are not Galilean invariant. On the other hand, their difference $\gamma_g[C_1 - C_2]$ (the grey area) is Galilean invariant. Note that $\gamma_g[C_1 - C_2]$ coincides with the Aharonov–Anandan phase γ_g^{AA} in this frame S when the area inside C_2 is zero.

under a Galilean transformation and according to our result (4) the difference of these two areas (in grey) is Galilean invariant.

Finally, we discuss an experimental situation for which the Galilean invariant structure of the geometric phase can be extracted from interference patterns. Suppose we have in the S frame the wavefunction

$$\Psi(x, 0) = \big(\psi^a(x, 0) + \psi^b(x, 0)\big)/\sqrt{2}.$$

We are considering an experiment in which both ψ^a and ψ^b undergo two different cyclic evolutions in the frame S; therefore ψ^a and ψ^b have each a diagram like that of Fig. 1. The first cycle results in

$$\Psi(x, t) = \big(e^{i\phi^a}\psi^a(x, 0) + e^{i\phi^b}\psi^b(x, 0)\big)/\sqrt{2}.$$

Separating the $\phi^{a,b}$ in its geometric and dynamical parts, the interference term for the probability density can be written as

$$I_1(x, t) = |\psi^a(x, 0)|\,|\psi^b(x, 0)| \times\cos\big(\Delta S(x, 0) + \Delta\gamma_{g1} + \Delta\gamma_{d1}\big), \quad (5)$$

with $\Delta S(x, 0) = S^b(x, 0) - S^a(x, 0)$ and $S^{a,b}$ the phases of $\psi^{a,b}$, $\Delta\gamma_{g1} = \gamma_g[C_1^b] - \gamma[C_1^a]$ and similarly for $\Delta\gamma_{d1}$. Maintaining all quantities fixed except $\Delta\gamma_{g1}$, the value of $\Delta\gamma_{g1}$ can be extracted from the interference pattern. At the end of a second cycle in $\mathscr{P}$ we can similarly extract $\Delta\gamma_{g2} = \gamma_g[C_2^b] - \gamma_g[C_2^a]$ from the new interference term

$$I_2(x, t') = |\psi^a(x, 0)|\,|\psi^b(x, 0)| \times\cos\big(\Delta S(x, 0) + \Delta\gamma_{g2} + \Delta\gamma_{d2}\big). \quad (6)$$

The interference terms in (5) and (6) relative to the $\tilde{S}$ frame are

$$\tilde{I}_1(\tilde{x}, t) = |\psi^a(\tilde{x}+vt, 0)|\,|\psi^b(\tilde{x}+vt, 0)| \times \cos\left(\Delta S(\tilde{x}+vt, 0) + \Delta\tilde{\gamma}_{g1} + F_a - F_b + \Delta\tilde{\gamma}_{d1}\right), \quad (7)$$

$$\tilde{I}_2(\tilde{x}, t') = |\psi^a(\tilde{x}+vt', 0)|\,|\psi^b(\tilde{x}+vt', 0)| \times \cos\left(\Delta S(\tilde{x}+vt', 0) + \Delta\tilde{\gamma}_{g2} + F_a - F_b + \Delta\tilde{\gamma}_{d2}\right). \quad (8)$$

From (4), the Galilean invariant quantities for this case are

$$\gamma_g[C_1^a - C_2^a] = \gamma_g[C_1^a] - \gamma_g[C_2^a], \quad (9)$$

$$\gamma_g[C_1^b - C_2^b] = \gamma_g[C_1^b] - \gamma_g[C_2^b]. \quad (10)$$

The experiment can extract from the interference patterns (5)–(8) the quantities

$$\Delta\gamma_g[C_1 - C_2] = \gamma_g[C_1^a - C_2^a] - \gamma_g[C_1^b - C_2^b] = \Delta\gamma_{g2} - \Delta\gamma_{g1}, \quad (11)$$

$$\Delta\gamma_g[\tilde{C}_1 - \tilde{C}_2] = \gamma_g[\tilde{C}_1^a - \tilde{C}_2^a] - \gamma_g[\tilde{C}_1^b - \tilde{C}_2^b] = \Delta\tilde{\gamma}_{g2} - \Delta\tilde{\gamma}_{g1} \quad (12)$$

and verify the Galilean invariant structure, $\Delta\gamma_g[\tilde{C}_1 - \tilde{C}_2] = \Delta\gamma_g[C_1 - C_2]$. This is in contrast to the non-Galilean invariant geometric phases $\Delta\gamma_{g1}$ and $\Delta\gamma_{g2}$; moreover $\Delta\tilde{\gamma}_{g1}$ and $\Delta\tilde{\gamma}_{g2}$ cannot be directly extracted separately as the corresponding interference patterns in (7) and (8) are also a consequence of the term $F_a - F_b$.

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