1. Introduction

In the discussion of random matrix theory and information theory, we basically explained the statistical aspect of ensembles of random matrices, The minimal "information content" of our ensemble was the first requirement we set, and we discussed it thoroughly. Now we discuss the second one, which was to find ensembles adequate to the symmetries that our system may have. We will see that generic symmetries restrict the kind of matrices we average over, leading to the definition of different ensembles.

2. Symmetries and Wigner's theorem

Our hypothesis was that a system's Hamiltonian may be modeled by a random matrix if the system is,loosely speaking, "complex enough". In principle, a random Hamiltonian is an Hermitian matrix with random entries (distributed according to some probability). However, there are certain generic symmetries that prevent some of the entries to have random values, so we have to be careful with what exactly we choose to model with a random matrix. If the system is invariant under parity, for example, we can use parity eigenstates as a basis for the Hamiltonian, which then splits it into two blocks in the diagonal:

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{1}$$

It is clear that the parity symmetry is imposing a very restrictive condition in the Hamiltionian (a lot of elements are zero), evident in this basis, but generally present in any basis. Therefore we don't expect to find generic properties of random matrices in this restricted (i.e. not-that-random) one. Nevertheless, we can always take H_+ or H_- separately, and look for generic properties in any of them.

Of course, this argument applies to whatever symmetry the system may have. If we now have rotational invariance, the Hamiltonian again splits in blocks caracterized by (J^2, J_z), and we have to choose one the blocks to look for generic properties. If there are no more symmetries, the blocks that remain can be considered to be random matrices. The lesson is clear: We expect generic properties to be found in each block after the partial diagonalization due to symmetry has been performed[1].

Then it seems that all we have to do is determine the generic properties of random Hermitian matrices, being aware that the don't necessarily represent the whole Hamiltonian. Is this the whole story? It would be, if all symmetries were represented by unitary operators. But recalling Wigner's theorem, we know that some perverse symmetries may be implemented by antiunitary operators. For our discussion, there will be only one of these symmetries, which is of course time reversal[2]. (Note that if time reversal is (strongly) violated, this *is* the whole story, and we are dealing with the symmetry class of random Hermitian matrices, called GUE, see below)

But what happens if T *is* a symmetry? The first difference with antiunitary operators is that the partial diagonalization doesn't work. Recall how to prove that the Hamiltonian matrix elements relating states with different eigenvalues of the parity operator P are orthogonal[3]. We just write:

$$\langle + | H | - \rangle = \left\langle + \left| P^\dagger P H P^\dagger P \right| - \right\rangle = - \langle + | H | - \rangle \tag{2}$$

And thus these elements are zero. Now we can try to do the same for time reversal. Remember the

[1] Note the following implication of this: In this discussion, a system either has a symmetry or it violates it strongly, as strongly as to mix all eigenstates of the symmetry operator with random matrix elements! We will elaborate more on this later

[2] There is at least another known case, charge conjugation in relativistic field theories. (Note that CPT=1, T antiunitary and P unitary forces C to be antiunitary) We won't treat this case for simplicity.

[3] There is an analogous proof for continuous symmetries, but this one is makes things clearer.

rather unusual properties of antiunitary operators (A good summary can be found in Messiah [3]). An antiunitary operator can be written as $T = UK_0$, where U is unitary, and K_0 is the complex conjugation operator. Under the time reversal map in Hilbert space, we have:

$$\mid b \rangle \to UK_0 \mid b \rangle \tag{3}$$
$$\langle a \mid \to (\langle a \mid K_0)U^\dagger \tag{4}$$
$$O \to TOT^\dagger = U(K_0 O K_0 U^\dagger) = UO^* U^\dagger \tag{5}$$
$$\langle a | O | b \rangle \to (\langle a | K_0) U^\dagger U O^* U^\dagger U K_0 | b \rangle = \langle a | O | b \rangle^* \tag{6}$$

So that matrix elements transform into their complex conjugates. (This is the peculiarity of antiunitary operators, note that they still conserve the modulus) Now try to apply the same argument as we did for parity. Because of complex conjugation, the identity 2 no more relates a matrix element with itself but with its complex conjugate. That implies that matrix elements between kets belonging to different representations of time reversal are not necessarily orthogonal, this is, there is no block diagonalization. In the end we will see that this not the important feature of time reversal, but it does say that there is something else going on. What is the restriction to be imposed in the matrix then? This question requires to discuss time reversal a little further.

3. Time reversal and rotational invariance

3.1. Time reversal representations

Let's discuss first the representations of time reversal. In general, both time reversal and parity are operations that leave the system invariant after applying them twice, because they are their own inverse. In a representation in Hilbert space this means that T^2 or P^2 are equal to a matrix of phases. In the case of parity this phases are irrelevant, for $P^2 = e^{i\phi} \to Pe^{i\phi/2}Pe^{i\phi/2} = 1$ and we may just redefine parity as $P' = Pe^{i\phi/2}$. However, once again due to antiunitarity, time reversal is different. First, the matrix of phases is restricted to have entries of ± 1, for if U is unitary $U^* U^T = 1$:

$$UK_0 UK_0 = e^{i\phi} \to UU^* = e^{i\phi} \tag{7}$$
$$U = e^{i\phi} U^T = e^{i\phi}(e^{i\phi} U^T)^T = e^{2i\phi} U \tag{8}$$

Which implies $e^{i\phi} = \pm 1$. Moreover, and this is the important part, when we have a minus, there is no possible redefinition of T that makes the phase irrelevant. $UK_0UK_0 = -1 = i^2 \to UiK_0U(-i)K_0 = 1 \neq T'T'$ simply due to the complex conjugation. The two cases, plus and minus, are physically different! Note that $UU^* = -1$ implies that U is at least two dimensional[4].

These two cases can be identified, in three dimensions, with integer and half integer spin, respectively. There is a simple way to understand this, which we now present. We could equally talk about total angular momentum, but clearly the distincition will come from spin, because orbital angular momentum is always integer. Note that we will use a particular representation of J and T to make things clear, but this identification holds independently of the representation. The key point in this discussion is that spin as an angular momentum must be reversed by time reversal.

We choose now the usual representation of the angular momentum operators. In the so called Condon and Shortley convention, J_+ and J_- are real matrices, and since $J_\pm = J_x \pm iJ_y$, J_y is pure imaginary and J_x

[4]Note that, in spite of the time reversal discrete group having the structure of $\mathcal{Z}_2$, which is abelian, this is not in contradiction with the fact that all unitary representations of abelian groups are one-dimensional, because this is a non unitary one.

is real, and so is J_z. In this case, we can see that time reversal (complex conjugating) is reverting only the y component. In order to get the full reversal of J as required, we need to add some unitary transformation U that reverts J_x and J_z, which is of course a rotation of π around the Y axis. This seems to mean that whenever we have spin, the matrix U in $T = UK_0$ is exactly this rotation.

Now it's easy to see the connection with $T^2 = \pm 1$. In the case of half integer spin, the rotation U is bivalued and gives a -1 when applied twice, so it corresponds to $T^2 = -1$. In the case of integer spin the rotation is faithful and $T^2 = 1$. So we see there is a connection between time reversal representations and spin. (Note a subtle implication here. If we associate the two time reversal representations with integer and half-integer total angular momentum, we realize that both representations are not allowed simultaneously in a Hamiltonian, because it either has integer or half-integer angular momentum.)

3.2. How does T invariance restrict H?

Let's now discuss the implications for the Hamiltonian when we have either of these cases. In the first one, $UU^* = 1$, is easy to prove that a change of basis B exists such that $BB^T = U$, and thus when we change basis the time reversal operator is just $T = K_0$. (This means that even though we chose J_y imaginary, there is a transformation that sets all J_i imaginary so no matrix U is required to rotate.) In this first case we realize that 5 reduces to complex conjugation, and therefore we see that invariance under time reversal just requires H to real. We note that in the second case there is not such transformation. We conclude then that *for integer spin time reversal invariant systems the Hamiltonian is real.* Moreover, in the case of rotational invariance, H commutes with all J_i. In the case $UU^* = 1$, $T = K_0$ and the Hamiltonian is real symmetric. But note that in the $UU^* = -1$, where U is a rotation generated by some J, H commutes with U due to rotational invariance, and time reversal is still just K_0 for the Hamiltonian. Thus this case reduces to previous one. *For time reversal and rotational invariant systems, the Hamiltonian is also real.*

Now think of the case of a spin 1/2 system. We can think of the Hamiltonian as made of two parts, one that couples to the spin and one that doesn't:

$$H = A \otimes I_{2x2} + B \otimes \sigma_1 + C \otimes \sigma_2 + D \otimes \sigma_3 \tag{9}$$

Where A,B,C,D are NxN matrices, Hermitian if H is Hermitian as we want. This is a convenient form of writing a $2Nx2N$ general Hermitian matrix. Now let's see how time reversal acts. It takes the complex conjugate and rotates in the spin space so that we have:

$$THT^\dagger = A^* \otimes I_{2x2} - B^* \otimes \sigma_1 - C^* \otimes \sigma_2 - D^* \otimes \sigma_3 \tag{10}$$

And we see that if time reversal is a symmetry, then $A = A^*$ is real symmetric but $B = -B^*$ is pure imaginary antisymmetric, and so are C and D. Thinking in terms of the $2Nx2N$ matrix, we have surely restricted it from being a totally random Hermitian matrix, but not to the point of being real. This is a case somewhere in between. This kind of matrix is called self dual in the context of quaternions, which we briefly describe now.

Quaternions are a generalization of complex numbers with three imaginary units that anticommute with each other, $\hat{\imath}, \hat{\jmath}, \hat{k}$, and square to -1: $q = a + b\hat{\imath} + c\hat{\jmath} + d\hat{k}$. We immediately realize that quaternions may be thought as linear combinations of $i\sigma_i$ and the identity. A dual quaternion is defined as reversing the sign of the "vector" part (b,c,d). Going back to our Hamiltonian, and taking out i from B,C,D, we see that H is matrix of quaternion real elements. We see clearly that the operation of time reversal may be seen as taking the dual of the quaternion matrix and conjugating the matrices A,B,C,D, and therefore a time reversal invariant H is called self dual real quaternion matrix. This is the third universality class. *For half integer spin, time reversal invariant systems with broken rotational symmetry, the Hamiltonian is a self dual real quaternion matrix.*(We have showed how the spin 1/2 case works, but the structure of a quaternion matrix may be proven to work for any half integer)

The ensemble of quaternion real self dual matrices is invariant under transformations called symplectic. In the lenguage of quaternions, these are quaternion matrices satisfying $S^D = S^\dagger = S^{-1}$, and are analogous to unitary matrices for Hermitian ones and orthogonal matrices for real symmetric ones.

4. Conclusions

The purpose of this discussion was to show that at the end of the day there are only three possible symmetry classes. After partial diagonalization due to generic symmetries has been performed, the blocks in the Hamiltonian belong to one of these three. The classes usually go by the name of the transformations that leave the ensemble invariant, so the ensemble of Hermitian matrices is rather called unitary, for example. Moreover, the ensembles are usually taken to have Gaussian distribution. These three ensembles are:

- If we don't have time reversal invariance, the Hamiltonian is Hermitian. Since the ensemble of all Hermitian matrices is invariant under a unitary transformation, we call this ensemble the Gaussian Unitary Ensemble (GUE).
- If we have time reversal symmetry and either a) integer total angular momentum (we may or may not have rotational invariance) or b) half integer total angular momentum and rotational invariance, the Hamiltonian is real symmetric. The invariance of the ensemble of real symmetric matrices under orthogonal transformations gives it the name of Gaussian Orthogonal Ensemble (GOE).
- If we have time reversal symmetry, no rotational invariance, and half integer total angular momentum, the Hamiltonian is quaternion real. Again, this ensemble is invariant under symplectic transformations, and so it is called the Gaussian Symplectic Ensemble (GSE).

These are the so called classical random matrix ensembles. Note that other ensembles are found when we include other symmetries (not that usual in quantum mechanics) such as charge conjugation or chiral symmetry. Seven more universality classes have been defined, four in the case of disordered superconductors, and three more in the case of disorder in the Dirac equation, the so called chiral classes. There is a very complete review in [4], and a more readable one in the appendix of [5].

It's very interesting to note that, although we have discussed the three ensembles in a rather physical fashion, there is a mathematical proof that the classification is exhaustive, if we are dealing with just time reversal and rotational invariance. This proof is related to the fact that there are only three division associative algebras over the reals: the reals themselves, the complex numbers, and the quaternions. Dyson's orignial paper [1] is the appropriate reference for this connection.

5. Summary

So what was the purpose of all this again? We were going to model a very complex Hamiltonian with a random matrix, looking for generic properties that H should share with almost all of the ensemble of random matrices, in the same sense as almost all the microstates for a given macrostate have the same physical properties in the thermodynamic limit. We have discussed in the first section what random means exactly in "ensemble of random matrices", and decided that basically any probability distribution is acceptable in the $N \to \infty$ limit. We chose Gaussian to make calculations easier.

We now have discussed how different symmetries lead to model a complex Hamiltonian with different ensembles. With the ensembles defined, we can now start to calculate what exactly are all these generic properties we have talked about, and see if all this is of any use for explaining experiments. And indeed it is! As we will see, the main prediction of random matrix theory is very easy to express: Fluctuation (or correlation) properties of eigenvalues of a random matrix are generic and just given by its symmetry class.

Referencias

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