UNIVERSITY OF Southampton

A "Grand" $\Delta(96)$ Flavour Symmetry and a Large Prediction for θ_{13}

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Invisibles Pre-Meeting

Based on Work in Progress with S.F. King and C. Luhn.

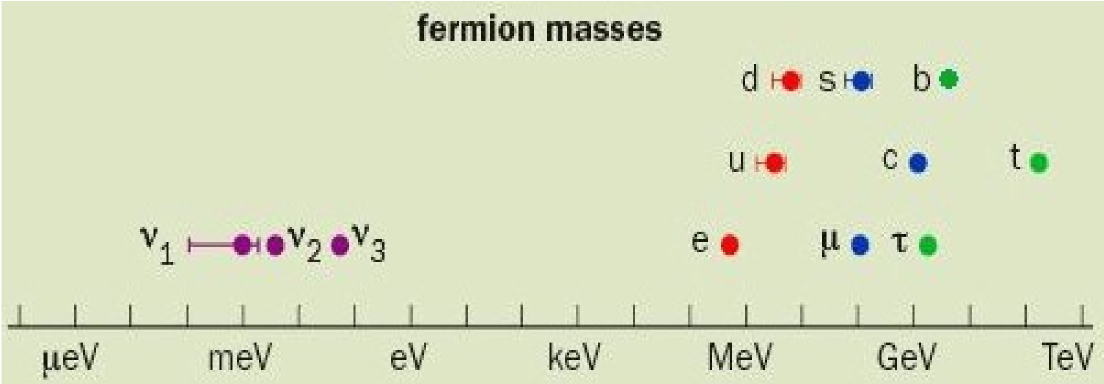
The Standard Model

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$

Triumph of modern science but incomplete-
fails to predict measured fermion masses and mixings

http://www.particleadventure.org/standard_model.html

What We Taste



Quark Mixing Angles

$$\mathcal{U}_{\rm CKM} = \mathcal{R}_1(\theta_{23}^{\rm CKM})\mathcal{R}_2(\theta_{13}^{\rm CKM}, \delta_{\rm CKM})\mathcal{R}_3(\theta_{12}^{\rm CKM})$$

$$\theta_{12}^{\rm CKM} = 13.0^\circ \pm 0.1^\circ$$

$$\theta_{23}^{\rm CKM} = 2.4^\circ \pm 0.1^\circ$$

$$\theta_{13}^{\rm CKM} = 0.2^\circ \pm 0.1^\circ$$

$$\delta_{\rm CKM} = 60^\circ \pm 14^\circ$$

Lepton Mixing Angles

$$\mathcal{U}_{\rm MNSP} = \mathcal{R}_1(\theta_\oplus)\mathcal{R}_2(\theta_{13}, \delta_{\rm MNSP})\mathcal{R}_3(\theta_\odot)\mathcal{P}$$

$$\theta_\odot = 34.4 \pm 1.0 \ \left(^{+3.2}_{-2.9}\right)^\circ$$

$$\theta_\oplus = 42.8^{+4.7}_{-2.9} \ \left(^{+10.7}_{-7.3}\right)^\circ$$

arXiv: 1106.4239

$$\theta_{13} = 8.8^o \pm 1.7^o$$

arXiv: 1203.1669

Adding Some Spice

(i.e. a discrete flavour symmetry that spontaneously broken by flavon field vevs to generate observed masses and mixings)

Try Adding Δ(96)

Shown by Toorop et al. in arXiv:1107.3486 and 1112.1340, to give a large leading order prediction of θ_{13} of about 12^o:

$$||U_{MNSP}|| = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}(\sqrt{3}+1) & 1 & \frac{1}{2}(\sqrt{3}-1) \\ 1 & 1 & 1 \\ \frac{1}{2}(\sqrt{3}-1) & 1 & \frac{1}{2}(\sqrt{3}+1) \end{pmatrix}$$

Notice atmospheric angle and solar angle equal~ 36.2^o . This is (about 2.4σ) below central value for the atmospheric angle, and solar is (about 2.4σ) above central value. (Fogli et al: arXiv 1106.6028)

Δ(96) has been used in models by G.J. Ding in arXiv:1201.3279

OUR GOAL: Construct a basis of Δ(96) which makes *explicit* connections with S, T, U, basis of S_4 (A_4) and use it to build a Grand Unified theory, but we must understand the group theory behind Δ(96) first.

$\Delta(6n^2)$

Class of SU(3) subgroups studied by Escobar and Luhn in arXiv: 0809.0639
Notice if n=4, we obtain Δ(96).

$$\Delta(96) \cong (Z_4 \otimes Z_4) \rtimes S_3$$

Complicated Group with 96 elements arranged into **10** conjugacy classes:

$$I \text{ (the trivial conjugacy class)}, 3C_4,\ 3C_2,\ 3C_4',\ 6C_4'',$$
$$32C_3,\ 12C_4''',\ 12C_8,\ 12C_2',\ \text{and}\ 12C_8'$$

Schoenflies notation- Subscript is order of elements in conjugacy class and number in front is # of elements in the class.

Use this information to calculate the irreducible representation of Δ(96):

$$1+3+3+3+6+32+12+12+12+12 = 96 = 1^2+1^2+2^2+\underbrace{3^2+3^2+3^2+3^2+3^2+3^2}_{\text{6 triplets!}}+6^2$$

Now that we have the conjugacy classes and irreps. What next?

Character Table of Δ(96)

As guided by arXiv: 0809.0639.

$\Delta(96)$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$	$\mathbf{3}$	$\tilde{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{3'}$	$\tilde{\mathbf{3}}'$	$\overline{\mathbf{3}}'$	$\mathbf{6}$
$\mathcal{I}$	1	1	2	3	3	3	3	3	3	6
$3C_4$	1	1	2	$-1+2i$	-1	$-1-2i$	$-1+2i$	-1	$-1-2i$	2
$3C_2$	1	1	2	-1	3	-1	-1	3	-1	-2
$3C_4'$	1	1	2	$-1-2i$	-1	$-1+2i$	$-1-2i$	-1	$-1+2i$	2
$6C_4''$	1	1	2	1	-1	1	1	-1	1	-2
$32C_3$	1	1	-1	0	0	0	0	0	0	0
$12C_2'$	1	-1	0	-1	-1	-1	1	1	1	0
$12C_8$	1	-1	0	i	1	$-i$	$-i$	-1	i	0
$12C_4'''$	1	-1	0	1	-1	1	-1	1	-1	0
$12C_8'$	1	-1	0	$-i$	1	i	i	-1	$-i$	0

Now that we have our character table, what next?

Kronecker Products of Δ(96)

$$1 \otimes x = x \text{ with } x \text{ any } \Delta(96) \text{ irrep}$$
$$1' \otimes 1' = 1$$
$$1' \otimes 2 = 2$$
$$1' \otimes r = r' \text{ when } r = 3,\ \tilde{3},\ \text{or } \overline{3}$$
$$1' \otimes r' = r \text{ when } r = 3,\ \tilde{3},\ \text{or } \overline{3}$$
$$1' \otimes 6 = 6$$
$$2 \otimes 2 = 1 \oplus 1' \oplus 2$$
$$2 \otimes r^m = r \oplus r' \text{ when } r = 3,\ \tilde{3},\ \text{or } \overline{3}$$
$$2 \otimes 6 = 6 \oplus 6$$
$$3^m \otimes 3^n = \tilde{3}^p \oplus \overline{3}' \oplus \overline{3}$$
$$3^m \otimes \tilde{3}^n = \overline{3}^p \oplus 6$$
$$3^m \otimes \overline{3}^n = 1^q \oplus 2 \oplus 6$$
$$\tilde{3}^m \otimes \tilde{3}^n = 1^q \oplus 2 \oplus \tilde{3} \oplus \tilde{3}'$$
$$\tilde{3}^m \otimes \overline{3}^n = 3^p \oplus 6$$
$$\overline{3}^m \otimes \overline{3}^n = 3 \oplus 3' \oplus \tilde{3}^p$$
$$3^m \otimes 6 = 3 \oplus \tilde{3} \oplus 3' \oplus \tilde{3}' \oplus 6$$
$$\tilde{3}^m \otimes 6 = 3 \oplus \overline{3} \oplus 3' \oplus \overline{3}' \oplus 6$$
$$\overline{3}^m \otimes 6 = \tilde{3} \oplus \overline{3} \oplus \tilde{3}' \oplus \overline{3}' \oplus 6$$
$$6 \otimes 6 = 1 \oplus 1' \oplus 2 \oplus 2 \oplus 3 \oplus 3' \oplus \tilde{3} \oplus \tilde{3}' \oplus \overline{3} \oplus \overline{3}' \oplus 6 \oplus 6$$

Here, 'm' and 'n' count the number of primes on their corresponding irreps. Furthermore p= “ ' ” if m+n is even and nothing if odd, and q= “ ' ” if m+n is odd and nothing if even.

$$\overline{3} \otimes \overline{3} = 3 \oplus 3' \oplus \tilde{3}'$$

$$3 \otimes 3 = \tilde{3}' \oplus \overline{3} \oplus \overline{3}'$$

$$\tilde{3} \otimes \tilde{3} = 1 \oplus 2 \oplus \tilde{3} \oplus \tilde{3}'$$

All very theoretical.....

Physicists like Explicit Representations

GOAL: Seek to calculate the explicit product decompositions of all tensor products of Δ(96). Start with presentation given to us by Escobar and Luhn:

$$\Delta(96) \cong (Z_4 \otimes Z_4) \rtimes S_3$$

$$a^3 = b^2 = (ab)^2 = c^4 = d^4 = 1$$

$$cd = dc$$

$$aca^{-1} = c^{-1}d^{-1} \qquad ada^{-1} = c$$

$$bcb^{-1} = d^{-1} \qquad bdb^{-1} = c^{-1}$$

Here, 'a' and 'b' generate the S_3 subgroup, and 'c' and 'd' the $Z_4 \otimes Z_4$ Abelian, normal subgroup.

Yet, we want to make a connection with the canonical s, t, and u generators....

Making the Connection

In Toorop et al, they relate a, b, c, and d to a smaller set of generators X and Y

$$a = Y^5 X Y^4 \quad b = X Y^2 X Y^5$$
$$c = X Y^2 X Y^4 \quad d = X Y^2 X Y^6$$

Multiplying various combination of a, b, c, d and their inverses yields:

$$Y = c^{-1} b = bd \qquad Y^2 = c^{-1} d \qquad X Y^5 = c a^{-1}$$
$$X = (c a^{-1})(c^{-1} b)(c^{-1} d) = d^{-1} c a^{-1} b c^{-1} d$$

Then another straightforward calculation reveals:

$$XY = ddca^{-1}dd \qquad XY^{-1}XY = ddac^{-1}dd$$

Which then reveals the new presentation for Δ(96) given in Toorop et al :

$$X^2 = Y^8 = (XY)^3 = (XY^{-1}XY)^3 = 1$$

A Caveat

$$X^2 = Y^8 = (XY)^3 = (XY^{-1}XY)^3 = 1$$

In Toorop et al., their explicit Y generator is diagonal. Yet, XY, the element associated with the low-energy symmetry of charged leptons, is not. We would like to work in a basis in which this is diagonal, and begin to relate our work to canonical s, t, and u generators.

Define $t = XY$ $u = X$. Then, apply unitary transformation that diagonalises 't' to 'u' and 's' to arrive at final representation.

$$u^2 = t^3 = (ut)^8 = (ut^{-1}ut)^3 = 1 \qquad s = u(ut)^4u(ut)^4$$

Notice 's' is *not* a generator in this particular presentation.

Thus, we know how to start from any Δ(96) representation given by Escobar and Luhn in 'a', 'b', 'c', and 'd' basis, and transform it into an 's', 't', and 'u', basis in which 't' is diagonal (via Toorop et al.) that can yield a large, nonzero value for θ_{13}.

HOWEVER, it turns out that if one tries to use this canonical S_4 basis for Δ(96), one has to permute e and μ in the basis to obtain a phenomenologically viable value for the reactor angle.

Δ(96) Representations

$$s_3 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

$$t_3 = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} \qquad \omega = e^{\frac{2\pi i}{3}}$$

$$u_3 = \frac{1}{3}\begin{pmatrix} -1+\sqrt{3} & -1-\sqrt{3} & -1 \\ -1-\sqrt{3} & -1 & -1+\sqrt{3} \\ -1 & -1+\sqrt{3} & -1-\sqrt{3} \end{pmatrix}$$

	s	t	u
$1:$	1	1	1
$1':$	1	1	-1
$2:$	$\mathcal{I}_{2\times 2}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$3:$	s_3	t_3	u_3
$\overline{3}:$	s_3	t_3^*	u_3
$3':$	s_3	t_3	$-u_3$
$\overline{3}':$	s_3	t_3^*	$-u_3$
$\tilde{3}:$	$\mathcal{I}_{3\times 3}$	t_3	vs_3
$\tilde{3}':$	$\mathcal{I}_{3\times 3}$	t_3	$-vs_3$
$6:$	$\begin{pmatrix} s_3 & 0 \\ 0 & s_3 \end{pmatrix}$	$\begin{pmatrix} t_3 & 0 \\ 0 & t_3 \end{pmatrix}$	$\begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix}$

$$w = \frac{1}{3}\begin{pmatrix} 1+i & 1+i & 1-2i \\ 1+i & 1-2i & 1+i \\ 1-2i & 1+i & 1+i \end{pmatrix} \qquad v = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Now that we have the generators for each irrep we can calculate CG-coefficients.

How to Build a Δ(96) SUSY GUT

With SU(5) GUT model building in mind, we want F~3, T~2, and T_3~1.
Want seesaw so introduce N~$\bar{3}$. We also Higgs fields uncharged under Δ(96)

Relevant Tensor Products (for now)

$$1 \otimes 1 = 1$$

$$2 \otimes 2 = 1 \oplus 1' \oplus 2 \qquad 2 \otimes 3 = 3 \oplus 3' \qquad 2 \otimes \bar{3} = \bar{3} \oplus \bar{3}'$$

$$3 \otimes 3 = \tilde{3}' \oplus \bar{3} \oplus \bar{3}' \qquad 3 \otimes \bar{3} = 1 \oplus 2 \oplus 6 \qquad \bar{3} \otimes \bar{3} = 3 \oplus 3' \oplus \tilde{3}'$$

We see that we get a renormalizable top quark mass, unsuppressed up and charm quark masses, and Dirac neutrino mass.

How can we fix these mostly problematic leading order predictions?

Completing the Recipe

Recall that we need flavon fields to couple to our mass terms and forbid (most) masses at renormalisable level.

“Trust” Δ(96) and add at least one flavon for each irreducible representation on the preceding slides' Kronecker products:

$$\phi^f_\rho$$

Superscript “f” represents up, down/charged lepton, and subscript “ρ” is a irreducible representation of Δ(96).

Notice, we also need an additional symmetry help forbid leading order contributions to the charm and up masses. Therefore, impose an additional U(1) symmetry.

A “Grand” Δ(96) Model

Field	T_3	T	F	N	H_5	$H_{\overline{5}}$	$H_{\overline{45}}$	ϕ_2^u	$\tilde{\phi}_2^u$	$\phi_{\overline{3}}^d$	$\tilde{\phi}_{\overline{3}}^d$	ϕ_2^d	$\phi_{\overline{3}'}^\nu$	$\phi_{\tilde{3}'}^\nu$
$SU(5)$	10	10	$\overline{5}$	1	5	$\overline{5}$	$\overline{45}$	1	1	1	1	1	1	1
$\Delta(96)$	1	2	3	$\overline{3}$	1	1	1	2	2	$\overline{3}$	$\overline{3}$	2	$\overline{3}'$	$\tilde{3}'$
$U(1)$	0	x	y	$-y$	0	0	z	$-2x$	0	$-y$	$-x-y-2z$	z	$2y$	$2y$

In the above, x, y, and z are 'carefully' chosen integers.

Model closely resembles/follows S_4xSU(5) model of Hagedorn, et al. in arXiv:1003.4249.

Notice that we have added a 45-dimensional Higgs in hope of obtaining the Georgi-Jarlkog relations for the charged lepton and down-type quark masses.

Up-Quark Sector

$$w_u = T_3 T_3 H_5 + \frac{1}{M} T T \phi_2^u H_5 + \frac{1}{M^2} T T \phi_2^u \tilde{\phi}_2^u H_5$$

'M' is generic messenger scale(presumably GUT) common to all higher dimensional operators. Order one couplings have been suppressed.

$$\langle \phi_2^u \rangle = \phi_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \langle \tilde{\phi}_2^u \rangle = \tilde{\phi}_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$M_u \approx v_u \begin{pmatrix} \tilde{\phi}_2^u \phi_2^u / M^2 & 0 & 0 \\ 0 & \phi_2^u / M & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi_2^u / M \approx \lambda^4 \qquad \tilde{\phi}_2^u / M \approx \lambda^4$$

$$\lambda \approx 0.22$$

$$m_u : m_c : m_t \approx \lambda^8 : \lambda^4 : 1$$

Down-Sector Masses and Mixings

$$w_d = \frac{1}{M} F T_3 \phi^d_{\bar{3}} H_{\bar{5}} + \frac{1}{M^2} (F \tilde{\phi}^d_{\bar{3}})_1 (T \phi^d_2)_1 H_{\overline{45}} + \frac{1}{M^3} ((F \phi^d_2 \phi^d_2)_3 (T \tilde{\phi}^d_3)_{\bar{3}})_1 H_{\bar{5}}$$

Notice we are choosing specific contractions due to Messenger Sector.

$$\langle \phi^d_2 \rangle = \phi^d_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \langle \tilde{\phi}^d_{\bar{3}} \rangle = \tilde{\phi}^d_{\bar{3}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \langle \phi^d_{\bar{3}} \rangle = \phi^d_{\bar{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

After electroweak symmetry breaking:

$$M_d \approx v_d \begin{pmatrix} 0 & (\phi^d_2)^2 \tilde{\phi}^d_{\bar{3}}/M^3 & (\phi^d_2)^2 \tilde{\phi}^d_{\bar{3}}/M^3 \\ (\phi^d_2)^2 \tilde{\phi}^d_{\bar{3}}/M^3 & \phi^d_2 \tilde{\phi}^d_{\bar{3}}/M^2 + (\phi^d_2)^2 \tilde{\phi}^d_{\bar{3}}/M^3 & (\phi^d_2)^2 \tilde{\phi}^d_{\bar{3}}/M^3 \\ 0 & 0 & \phi^d_{\bar{3}}/M \end{pmatrix}$$

$$\phi^d_2/M \approx \lambda \qquad \phi^d_{\bar{3}}/M \approx \lambda^{1+k} \qquad \tilde{\phi}^d_{\bar{3}}/M \approx \lambda^{2+k} \qquad k = 0,1$$

Unlike up-sector:

$$\theta^d_{13} \approx \lambda^3 \qquad \theta^d_{12} \approx \lambda \qquad \theta^d_{23} \approx \lambda^2$$

$$m_d : m_s : m_b \approx \lambda^4 : \lambda^2 : 1$$

Charged Lepton Sector

$$M_e \approx v_d \begin{pmatrix} 0 & (\phi_2^d)^2 \tilde{\phi}_3^d / M^3 & 0 \\ (\phi_2^d)^2 \tilde{\phi}_3^d / M^3 & -3\phi_2^d \tilde{\phi}_3^d / M^2 + (\phi_2^d)^2 \tilde{\phi}_3^d / M^3 & 0 \\ (\phi_2^d)^2 \tilde{\phi}_3^d / M^3 & (\phi_2^d)^2 \tilde{\phi}_3^d / M^3 & \phi_3^d / M \end{pmatrix}$$

Notice the -3 on the (22) entry from 45-dimensional Higgs coupling to give Georgi-Jarlskog Relation: $m_d = 3\mathrm{m}_e, m_s = m_\mu/3, \quad m_b = m_\tau$

$$m_e : m_\mu : m_\tau \quad \approx \quad (1/3)\lambda^4 : 3\lambda^2 : 1$$

Also obtain the Gatto-Sartori-Tonin relation:

$$\theta_{12}^e \approx \theta_{12}^d \approx \sqrt{m_d/m_s}$$

Furthermore, we receive a non-trivial mixing from this sector which will shift the prediction from the neutrino sector by several degrees:

$$\theta_{12}^e \approx (1/3)\lambda \qquad \theta_{23}^e \approx 0 \qquad \theta_{13}^e \approx 0$$

k=0 implies tanβ~30 and and k=1 implies tanβ~5 because

$$m_b \approx m_\tau \approx \lambda^{1+k} v_d$$

Neutrino Sector

$$w_\nu = y_D F N H_5 + N N \phi^\nu_{\bar{3}'} + N N \phi^\nu_{\tilde{3}'}$$

Dirac Mass Matrix:

$$M_D = y_D v_d \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Majorana Mass Matrix requires flavons allignments which preserve $Z_2^s \otimes Z_2^u$ low-energy subgroup:

$$\langle \phi^\nu_{\bar{3}'} \rangle = \phi^\nu_{\bar{3}'} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \langle \phi^\nu_{\tilde{3}'} \rangle = \phi^\nu_{\tilde{3}'} \begin{pmatrix} v_1 \\ \frac{1}{2}(v_1 + v_3) \\ v_3 \end{pmatrix}$$

$$M_{Maj} = \phi^\nu_{\bar{3}'} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} + \phi^\nu_{\tilde{3}'} \begin{pmatrix} v_3 & v_1 & \frac{1}{2}(v_1 + v_3) \\ v_1 & \frac{1}{2}(v_1 + v_3) & v_3 \\ \frac{1}{2}(v_1 + v_3) & v_3 & v_1 \end{pmatrix}$$

Of course we want a seesaw: $m_\nu = M_D M_{Maj}^{-1} M_D^T = y_D^2 v_D^2 M_{Maj}^{-1}$

Neutrinos (II)

After forming the low mass neutrino matrix from the Seesaw Mechanism, we must diagonalise to reveal the neutrino masses and mixings:

$$m_\nu = U_\nu^* Diag(m_1, m_2, m_3) U_\nu^\dagger$$

$$m_1 = \frac{2y_D^2 v_u^2}{\sqrt{3}\phi^\nu_{3'}(v_1 - v_3) - 6\phi^\nu_{3'}} \qquad m_2 = \frac{2y_D^2 v_u^2}{3\phi^\nu_{3'}(v_1 + v_3)} \qquad m_3 = \frac{2y_D^2 v_u^2}{\sqrt{3}\phi^\nu_{3'}(v_3 - v_1) - 6\phi^\nu_{3'}}$$

$$U_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}(1+\sqrt{3}) & 1 & \frac{1}{2}(\sqrt{3}-1) \\ -1 & 1 & 1 \\ \frac{1}{2}(1-\sqrt{3}) & 1 & \frac{1}{2}(-1-\sqrt{3}) \end{pmatrix}$$

$$\theta^\nu_{12} = \tan^{-1}\left(\frac{2\sqrt{3}}{3+\sqrt{3}}\right) \approx 36.2^\circ \qquad \theta^\nu_{13} \approx \frac{1}{6}(3-\sqrt{3}) \approx 12.1^\circ \quad |\theta^\nu_{23}| = \left|\tan^{-1}\left(\frac{\sqrt{3}-3}{\sqrt{3}}\right)\right| \approx 36.2^\circ$$

As desired! Need to shift values closer to measured values with Charged Lepton corrections, RGE's, etc...

Conclusion

- Daya Bay has measured the Reactor Neutrino Mixing Angle, and it is large.
- Certain groups, like $\Delta(96)$, can predict a large “leading order” value for θ_{13} that only needs to be shifted by a few degrees to obtain the measured value.
- We have constructed an SU(5) SUSY GUT with $\Delta(96)$ flavour symmetry(justified vacuum alignment to appear in final paper), in a basis which explicitly draws analogy to existing flavour symmetries' generator structures.
- This has only been possible because it is an exciting time to be in particle physics!